1.
$$f(\theta) = 4\cos^2\theta - 3\sin^2\theta$$

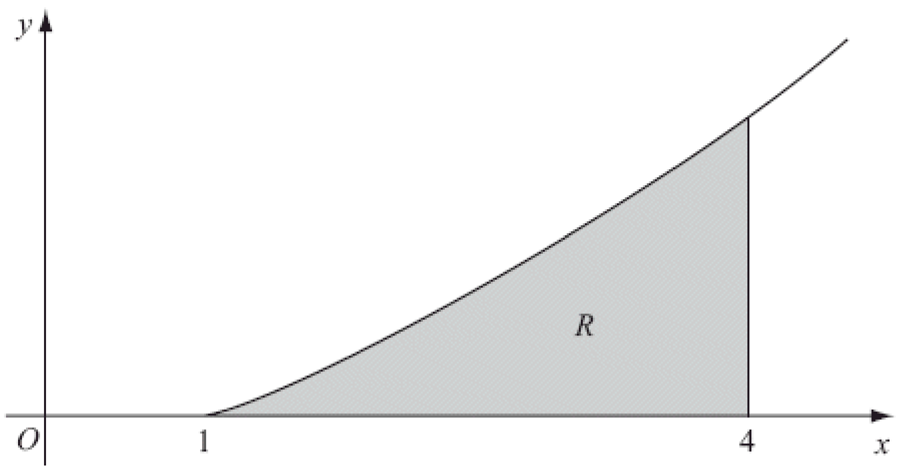
(a) Show that $f(\theta) = \frac{1}{2} + \frac{7}{2}\cos 2\theta$.

(3)

(b) Hence, using calculus, find the exact value of $\int_0^{\frac{\pi}{2}} \theta\, f(\theta)\, d\theta$

(7)
(Total 10 marks)

2.



The diagram above shows a sketch of the curve with equation $y = x \ln x$, $x \geq 1$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 4$.

The table shows corresponding values of x and y for $y = x \ln x$.

x	1	1.5	2	2.5	3	3.5	4
y	0	0.608			3.296	4.385	5.545

(a) Complete the table with the values of y corresponding to $x = 2$ and $x = 2.5$, giving your answers to 3 decimal places.

(2)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

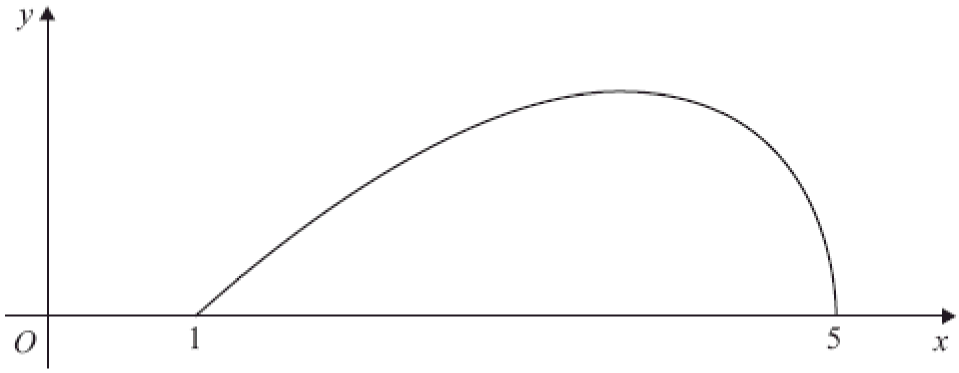
(4)

(c) (i) Use integration by parts to find $\int x \ln x \, dx$.

(ii) Hence find the exact area of R, giving your answer in the form $\frac{1}{4}(a \ln 2 + b)$, where a and b are integers.

(7)
(Total 13 marks

3. (a) Find $\int \sqrt{(5-x)} \, dx$



(2)

The diagram above shows a sketch of the curve with equation

$$y = (x - 1)\sqrt{(5 - x)}, \qquad 1 \le x \le 5$$

(b) (i) Using integration by parts, or otherwise, find

$$\int (x-1)\sqrt{(5-x)} \, dx$$

(4)

(ii) Hence find $\int_1^5 (x-1)\sqrt{(5-x)}\,dx$.

(2)
(Total 8 marks)

4. (a) Find $\int \tan^2 x\,dx$.

(2)

(b) Use integration by parts to find $\int \frac{1}{x^3} \ln x\,dx$.

(4)

(c) Use the substitution $u = 1 + e^x$ to show that

$$\int \frac{e^{3x}}{1+e^x}\,dx = \frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + k,$$

where k is a constant.

(7)
(Total 13 marks)

5. (a) Use integration by parts to find $\int xe^x\,dx$.

(3)

(b) Hence find $\int x^2e^x\,dx$.

(3)
(Total 6 marks)

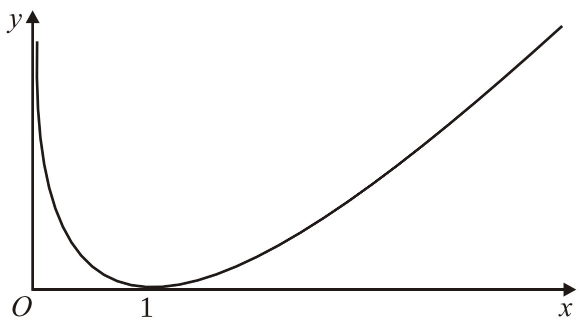
6. (a) Find $\int x\cos 2x\,dx$. .

(4)

(b) Hence, using the identity $\cos 2x = 2\cos^2 x - 1$, deduce $\int x\cos^2 x \, dx$.

(3)
(Total 7 marks)

7.



The figure above shows a sketch of the curve with equation $y = (x - 1) \ln x$, $x > 0$.

(a) Complete the table with the values of y corresponding to $x = 1.5$ and $x = 2.5$.

x	1	1.5	2	2.5	3
y	0		ln 2		2ln 3

(1)

Given that $I = \int_1^3 (x-1)\ln x \, dx$

(b) use the trapezium rule

(i) with values of y at $x = 1$, 2 and 3 to find an approximate value for I to 4 significant figures,

(ii) with values of y at $x = 1$, 1.5, 2, 2.5 and 3 to find another approximate value for I to 4 significant figures.

(5)

(c) Explain, with reference to the figure above, why an increase in the number of values improves the accuracy of the approximation.

(1)

(d) Show, by integration, that the exact value of $\int_1^3 (x-1)\ln x \, dx$ is $\frac{3}{2}\ln 3$.

(6)
(Total 13 marks)

8. $$f(x) = (x^2 + 1) \ln x, \qquad x > 0.$$

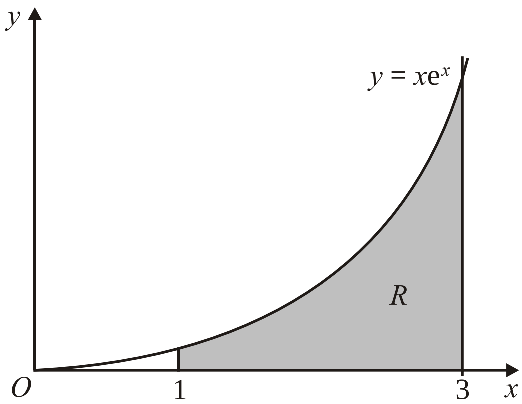
(a) Use differentiation to find the value of $f'(x)$ at x = e, leaving your answer in terms of e.

(4)

(b) Find the exact value of $\int_1^e f(x)\, dx$

(5)
(Total 9 marks)

9.



The figure above shows the finite shaded region, R, which is bounded by the curve $y = xe^x$, the line $x = 1$, the line $x = 3$ and the x-axis.

The region R is rotated through 360 degrees about the x-axis.

Use integration by parts to find an exact value for the **volume** of the solid generated.

(Total 8 marks)

10. (a) Use integration by parts to find

$$\int x\cos 2x \, \mathrm{d}x \, .$$

(4)

(b) Hence, or otherwise, find

$$\int x\cos^2 x \, \mathrm{d}x \, .$$

(3)
(Total 7 marks)

11. (a) Use integration by parts to show that

$$\int x \operatorname{cosec}^2\left(x+\frac{\pi}{6}\right)\mathrm{d}x = -x\cot\left(x+\frac{\pi}{6}\right) + \ln\left[\sin\left(x+\frac{\pi}{6}\right)\right] + c, \qquad -\frac{\pi}{6} < x < \frac{\pi}{3}.$$

(3)

(b) Solve the differential equation

$$\sin^2\left(x+\frac{\pi}{6}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy(y+1)$$

to show that $\frac{1}{2}\ln\left|\frac{y}{y+1}\right| = -x\cot\left(x+\frac{\pi}{6}\right) + \ln\left[\sin\left(x+\frac{\pi}{6}\right)\right] + c.$

(6)

Given that $y = 1$ when $x = 0$,

(c) find the exact value of y when $x = \frac{\pi}{12}$.

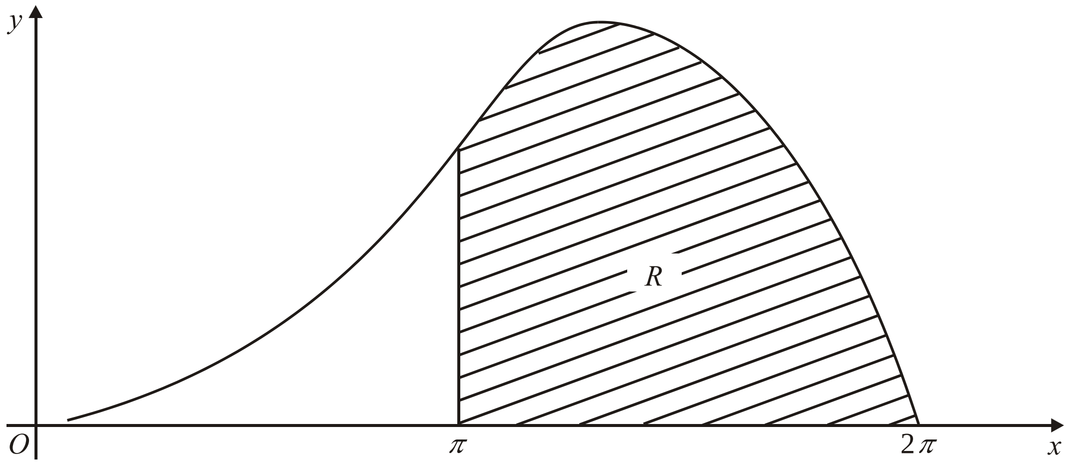
(6)
(Total 15 marks)

12. Given that $y = 1$ at $x = \pi$, solve the differential equation

$$\frac{dy}{dx} = yx^2 \cos x, \quad y > 0.$$

(Total 9 marks)

13.



The diagram above shows the curve with equation

$$y = x^2 \sin\left(\tfrac{1}{2}x\right), \qquad 0 < x \leq 2\pi.$$

The finite region R bounded by the line $x = \pi$, the x-axis, and the curve is shown shaded in Fig 1.

(a) Find the exact value of the area of R, by integration. Give your answer in terms of π.

(7)

The table shows corresponding values of x and y.

x	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y	9.8696	14.247	15.702	G	0

(b) Find the value of G.

(1)

(c) Use the trapezium rule with values of $x^2 \sin\left(\frac{1}{2}x\right)$

(i) at $x = \pi$, $x = \dfrac{3\pi}{2}$ and $x = 2\pi$ to find an approximate value for the area R, giving your answer to 4 significant figures,

(ii) at $x = \pi$, $x = \dfrac{5\pi}{4}$, $x = \dfrac{3\pi}{2}$, $x = \dfrac{7\pi}{4}$ and $x = 2\pi$ to find an improved approximation for the area R, giving your answer to 4 significant figures.

(5)
(Total 13 marks)

1.	(a)	$\mathrm{f}(\theta) = 4\cos^2\theta - 3\sin^2\theta$			
		$= 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) - 3\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)$		M1 M1	
		$= \frac{1}{2} + \frac{7}{2}\cos 2\theta \quad *$	cso	A1	3
	(b)	$\int \theta\cos 2\theta\,\mathrm{d}\theta = \frac{1}{2}\theta\sin 2\theta - \frac{1}{2}\int \sin 2\theta\,\mathrm{d}\theta$		M1 A1	
		$= \frac{1}{2}\theta\sin 2\theta + \frac{1}{4}\cos 2\theta$		A1	
		$\int \theta\,\mathrm{f}(\theta)\,\mathrm{d}\theta = \frac{1}{4}\theta^2 + \frac{7}{4}\theta\sin 2\theta + \frac{7}{8}\cos 2\theta$		M1 A1	
		$\left[\ \ldots\ \right]_0^{\frac{\pi}{2}} = \left[\frac{\pi^2}{16} + 0 - \frac{7}{8}\right] - \left[0 + 0 + \frac{7}{8}\right]$		M1	
		$= \frac{\pi^2}{16} - \frac{7}{4}$		A1	7

[10]

2.	(a)	1.386, 2.291	awrt 1.386, 2.291	B1 B1	2
	(b)	$A \approx \frac{1}{2} \times 0.5(\ldots)$		B1	
		= ... (0+2(0.608+1.386+2.291+3.296 +4.385)+5.545)		M1	
		= 0.25(0+2(0.608+1.386+2.291+3.296 +4.385)+5.545)	ft their (a)	A1ft	
		=0.25×29.477 ... ≈7.37	cao	A1	4
	(c)	(i) $\int x\ln x\,\mathrm{d}x = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x}\,\mathrm{d}x$		M1 A1	
		$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,\mathrm{d}x$			
		$= \frac{x^2}{2}\ln x - \frac{x^2}{4}(+C)$		M1 A1	
		(ii) $\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^4 = (8\ln 4 - 4) - \left(-\frac{1}{4}\right)$		M1	

$=8\ln 4 - \frac{15}{4}$

$=8(2\ln 2) - \frac{15}{4}$ ln 4 = 2ln 2 seen or implied M1

$=\frac{1}{4}(64\ln 2 - 15)$ $a = 64, b = -15$ A1 7

[13]

3. (a) $\int\sqrt{(5-x)}\,dx = \int(5-x)^{\frac{1}{2}}\,dx = \frac{(5-x)^{\frac{1}{2}}}{-\frac{3}{2}}(+C)$ M1 A1 2

$\left(= -\frac{2}{3}(5-x)^{\frac{3}{2}} + C\right)$

(b) (i) $\int(x-1)\sqrt{(5-x)}\,dx = -\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} + \frac{2}{3}\int(5-x)^{\frac{3}{2}}\,dx$ M1 A1ft

$= \quad \ldots \quad + \frac{2}{3}\times\frac{(5-x)^{\frac{5}{2}}}{-\frac{5}{2}}(+C)$ M1

$-\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{4}{15}(5-x)^{\frac{5}{2}}\ (+C)$ A1 4

(ii) $\left[-\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{4}{15}(5-x)^{\frac{5}{2}}\right]_1^5 = (0-0) - \left(0 - \frac{4}{15}\times 4^{\frac{5}{2}}\right)$

$= \frac{128}{15}\left(= 8\frac{8}{15} \approx 8.53\right)$ awrt 8.53 M1 A1 2

Alternatives for (b)

$u^2 = 5 - x \Rightarrow 2u\frac{du}{dx} = -1\left(\Rightarrow \frac{dx}{du} = -2u\right)$

$\int(x-1)\sqrt{(5-x)}\,dx = \int(4-u^2)u\frac{dx}{du}\,du = \int(4-u^2)\,u\,(-2u)\,du$ M1 A1

$= \int(2u^4 - 8u^2)\,du = \frac{2}{5}u^5 - \frac{8}{3}u^3(+C)$ M1

$= \frac{2}{5}(5-x)^{\frac{5}{2}} - \frac{8}{3}(5-x)^{\frac{3}{2}}\ (+C)$ A1

Alternatives for (c)

$x = 1 \Rightarrow u = 2,\ x = 5 \Rightarrow u = 0$

$$\left[\frac{2}{5}u^5 - \frac{8}{3}u^3\right]_2^0 = (0-0) - \left(\frac{64}{5} - \frac{64}{3}\right)$$ M1

$$= \frac{128}{15}\left(= 8\frac{8}{15} \approx 8.53\right)$$ awrt 8.53 A1 2

[8]

4. (a) $\int \tan^2 x\,dx$

[*NB* : <u>$\sec^2 A = 1 + \tan^2 A$</u> gives <u>$\tan^2 A = \sec^2 A - 1$</u>] The correct <u>underlined identity</u>. M1 oe

$$= \int \sec^2 x - 1\,dx$$

$= \underline{\tan x - x (+c)}$ Correct integration with/without + c A1 2

(b) $\int \frac{1}{x^3}\ln x\,dx$

$$\left\{\begin{array}{l} u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \\ \frac{dv}{dx} = x^{-3} \Rightarrow v = \frac{x^{-2}}{-2} = \frac{-1}{2x^2} \end{array}\right\}$$

$$= -\frac{1}{2x^2}\ln x - \int -\frac{1}{2x^2}.\frac{1}{x}\,dx$$ Use of 'integration by parts' formula in the correct direction. M1

Correct direction means that $u = \ln x$.

Correct expression. A1

$$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\int \frac{1}{x^3}\,dx$$ An attempt to multiply through $\frac{k}{x^n}, n \in$ □ n . . .2 by $\frac{1}{x}$ and an attempt to ...

$$= \underline{-\frac{1}{2x^2}\ln x + \frac{1}{2}\left(-\frac{1}{2x^2}\right)} (+c)$$ ... "integrate"(process the result); M1

<u>correct solution</u> with/without + c A1 oe 4

(c) $\int \frac{e^{3x}}{1+e^x}\,dx$

$\left\{u = 1+e^x \Rightarrow \underline{\frac{du}{dx} = e^x}, \underline{\frac{dx}{du} = \frac{1}{e^x}}, \underline{\frac{dx}{du} = \frac{1}{u-1}}\right\}$ Differentiating to find any one of the <u>three underlined</u> B1

$\int \frac{e^{2x}.e^x}{1+e^x}\,dx = \int \frac{(u-1)^2.e^x}{u}.\frac{1}{e^x}\,du$ Attempt to substitute for $e^{2x} = f(u)$, their $\frac{dx}{du} = \frac{1}{e^x}$ and $u = 1+e^x$

or = $\int \frac{(u-1)^3}{u}.\frac{1}{(u-1)}\,du$ or $e^{3x} = f(u)$, their $\frac{dx}{du} = \frac{1}{u-1}$ and $u = 1 + e^x$. M1 *

= $\int \frac{(u-1)^2}{u}\,du$ $\underline{\int \frac{(u-1)^2}{u}\,du}$ A1

= $\int \frac{u^2-2u+1}{u}\,du$ An attempt to multiply out their numerator to give at least three terms

= $\int u - 2 + \frac{1}{u}\,du$ and divide through each term by u dM1 *

= $\frac{u^2}{2} - 2u + \ln u\ (+c)$ Correct integration with/without +c A1

= $\frac{(1+e^x)^2}{2} - 2(1+e^x) + \ln(1+e^x) + c$ Substitutes $u = 1 + e^x$ back into their integrated expression with at least two terms. dM1 *

= $\frac{1}{2} + e^x + \frac{1}{2}e^{2x} - 2 - 2e^x + \ln(1+e^x) + c$

= $\frac{1}{2} + e^x + \frac{1}{2}e^{2x} - 2 - 2e^x + \ln(1+e^x) + c$

= $\frac{1}{2}e^{2x} - e^x + \ln(1+e^x) - \frac{3}{2} + c$

= $\frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + k$ **AG** $\underline{\frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + k}$ must use a + c + and "$-\frac{3}{2}$" combined. A1 cso 7

[13]

5. (a) $\left\{\begin{array}{l} u = x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1 \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x \end{array}\right\}$

$\int x\mathrm{e}^x \mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x .1\mathrm{d}x$

$= x\mathrm{e}^x - \int \mathrm{e}^x \mathrm{d}x$

$= x\ \mathrm{e}^x - \mathrm{e}^x\ (+\ c)$

Use of 'integration by parts' formula in the **correct direction**. (See note.)	M1	
Correct expression. (Ignore dx)	A1	
Correct integration with/without + c	A1	3

Note integration by parts in the **correct direction** means that u and $\dfrac{\mathrm{d}v}{\mathrm{d}x}$ must be assigned/used as $u = x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$ in this part for example.

+ c is not required in this part.

(b) $\left\{\begin{array}{l} u = x^2 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x \Rightarrow v = \mathrm{e}^x \end{array}\right\}$

$\int x^2\mathrm{e}^x \mathrm{d}x = x^2\mathrm{e}^x - \int \mathrm{e}^x .2x\mathrm{d}x$

$= x^2\mathrm{e}^x - 2\int x\mathrm{e}^x \mathrm{d}x$

$= x^2\mathrm{e}^x - 2(x\mathrm{e}^x - \mathrm{e}^x) + c$

$\left\{\begin{array}{l} = x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + c \\ = \mathrm{e}^x(x^2 - 2x + 2) + c \end{array}\right\}$

Use of 'integration by parts' formula in the **correct direction**.	M1	
Correct expression. (Ignore dx)	A1	
Correct expression **including + c**. **(seen at any stage!)** You can ignore subsequent working.	A1 **ISW**	3

Ignore subsequent working

+ c is required in this part.

[6]

6. (a) $\left\{\begin{array}{ll} u = x & \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 1 \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x & \Rightarrow v = \dfrac{1}{2}\sin 2x \end{array}\right\}$

$$\text{Int} = \int x\cos 2x\mathrm{d}x = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x.1\mathrm{d}x$$

$$= \frac{1}{2}x\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + c$$ 4

$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

M1 **(*see note below*)**
Use of 'integration by parts' formula in the correct direction.

A1 Correct expression.

dM1 $\sin 2x \to -\dfrac{1}{2}\cos 2x$

or $\sin kx \to -\dfrac{1}{k}\cos kx$

with $k \neq 1$, $k > 0$

A1 Correct expression with + c

(b) $\int x\cos^2 x\,dx = \int x\left(\frac{\cos 2x+1}{2}\right)dx$

$= \frac{1}{2}\int x\cos 2x\,dx + \frac{1}{2}\int x\,dx$

$= \underline{\frac{1}{2}\left(\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)};+\frac{1}{2}\int x\,dx$ 3

$= \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + \frac{1}{4}x^2\ (+c)$

M1 Substitutes correctly for $\cos^2 x$ in the given integral

A1ft $\frac{1}{2}$(their answer to (a)); or underlined expression

A1 Completely correct expression with/without $+c$

Notes:

Int = $\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x \pm \int \frac{1}{2}\sin 2x.1\,dx$

M1 This is acceptable for M1

$$\begin{Bmatrix} u = x & \Rightarrow \frac{du}{dx} = 1 \\ \frac{dv}{dx} = \cos 2x \Rightarrow v = \lambda \sin 2x & \end{Bmatrix}$$

Int = $\int x\cos 2x\,dx = \lambda x\sin 2x \pm \int \lambda\sin 2x.1\,dx$

M1 This is also acceptable for M1

Aliter (b) Way 2

$\int x\cos^2 x\,dx = \int x\left(\frac{\cos 2x+1}{2}\right)dx$

$$\begin{Bmatrix} u = x & \Rightarrow \frac{du}{dx} = 1 \\ \frac{dv}{dx} = \frac{1}{2}\cos 2x + \frac{1}{2} \Rightarrow v = \frac{1}{4}\sin 2x + \frac{1}{2}x & \end{Bmatrix}$$

$= \frac{1}{4}x\sin 2x + \frac{1}{2}x^2 - \int\left(\frac{1}{4}\sin 2x + \frac{1}{2}x\right)dx$

$= \underline{\frac{1}{4}x\sin 2x} + \frac{1}{2}x^2 + \underline{\frac{1}{8}\cos 2x} - \frac{1}{4}x^2 + c$

$= \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + \frac{1}{4}x^2\ (+c)$

M1 Substitutes **<u>correctly</u>** for $\cos^2 x$ in the given integral...
...or
$u = x$ and $\frac{dv}{dx} = \frac{1}{2}\cos 2x + \frac{1}{2}$

A1ft $\frac{1}{2}$ (their answer to (a)); or underlined expression

A1 Completely correct expression with/without + c

Aliter (b) Way 3

$$\int x\cos 2x\,dx = \int x(2\cos^2 x - 1)\,dx$$

$$\Rightarrow 2\int x\cos^2 x\,dx - \int x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

$$\Rightarrow \int x\cos^2 x\,dx = \underline{\frac{1}{2}\left(\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)}; + \frac{1}{2}\int x\,dx$$

$$= \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + \frac{1}{4}x^2 (+c)$$

M1 Substitutes **correctly** for $\cos 2x$ in $\int x\cos 2x\,dx$

A1ft $\frac{1}{2}$ (their answer to (a)); or underlined expression

A1 Completely correct expression with/without + c

[7]

7. (a)

x	1	1.5	2	2.5	3
y	0	0.5 ln 1.5	ln 2	1.5 ln 2.5	2 ln 3
or y	0	0.2027325541 ...	ln2	1.374436098 ...	2 ln 3

Either 0.5 ln 1.5 and 1.5 ln 2.5 B1
or awrt 0.20 and 1.37
(or mixture of decimals and ln's)

(b) (ii) $l_1 \approx \frac{1}{2}\times 1 \times \{\underline{0 + 2(\ln 2) + 2\ln 3}\}$ M1;

For structure of trapezium rule {…………};

$\frac{1}{2}\times$ 3.583518938... = 1.791759... = 1.792 (4sf) A1 cao

(ii) $l_2 \approx \frac{1}{2}\times 0.5; \times \{\underline{0 + 2(0.5\ln 1.5 + \ln 2 + 1.5\ln 2.5) + 2\ln 3}\}$

Outside brackets $\frac{1}{2}$ × 0.5 B1

For structure of trapezium rule {…………..}; M1 ft

$= \frac{1}{4}\times$ 6.737856242....=1.684464 A1 5 *awrt 1.684*

(c) With increasing ordinates, the line segments at the top of the trapezia are closer to the curve. B1 1

Reason or an appropriate diagram elaborating the correct reason.

(d) $\left\{\begin{array}{ll} u = \ln x & \Rightarrow \frac{du}{dx} = \frac{1}{x} \\ \frac{dv}{dx} = x - 1 & \Rightarrow v = \frac{x^2}{2} - x \end{array}\right\}$ M1

Use of 'integration by parts' formula in the correct direction

$$\mathrm{I} = \left(\frac{x^2}{2} - x\right)\ln x - \int \frac{1}{x}\left(\frac{x^2}{2} - x\right)dx$$ A1

Correct expression

$$= \left(\frac{x^2}{2} - x\right)\ln x - \underline{\int \left(\frac{x}{2} - x\right)dx}$$

An attempt to multiply at least one term through by $\frac{1}{x}$ and an attempt to ...

$$= \left(\frac{x^2}{2} - x\right)\ln x - \underline{\left(\frac{x^2}{4} - x\right)}(+c)$$

... integrate; M1
correct integration A1

$$\therefore \mathrm{I} = \left[\left(\frac{x^2}{2} - x\right)\ln x - \frac{x^2}{4} + x\right]_1^3$$

$$= \left(\tfrac{3}{2}\ln 3 - \tfrac{9}{4} + 3\right) - \left(-\tfrac{1}{2}\ln 1 - \tfrac{1}{4} + 1\right)$$ ddM1

Substitutes limits of 3 and 1 and subtracts.

$$= \tfrac{3}{2}\ln 3 + \tfrac{3}{4} + 0 - \tfrac{3}{4} = \underline{\tfrac{3}{2}\ln 3}\ \ \mathbf{AG}$$ A1 cso 6

Aliter Way 2

(d) $\int (x-1)\ln x\,dx = \int x\ln x\,dx - \int \ln x\,dx$

$$\int x\ln x = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}.\left(\frac{1}{x}\right)dx\ \mathrm{J}$$ M1

Correct application of 'by parts'

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4}(+c)$$ A1

Correct integration

$\int \ln x \, dx = x \ln x - \int x.\left(\frac{1}{x}\right) dx$ M1

Correct application of 'by parts''

$= x \ln x - x\,(+c)$ A1

Correct integration

$\therefore \int_1^3 (x-1)\ln x \, dx = \left(\frac{9}{2}\ln 3 - 2\right) - \left(3\ln 3 - 2\right) = \frac{3}{2}\ln 3$ **AG**

Substitutes limits of 3 and 1 into both integrands and subtracts. dd M1

$\frac{3}{2}\ln 3$ A1 cso 6

Aliter Way 3

(d) $\left\{\begin{array}{ll} u = \ln x & \Rightarrow \frac{du}{dx} = \frac{1}{x} \\ \frac{dv}{dx} = x - 1 & \Rightarrow v = \frac{(x-1)^2}{2} \end{array}\right\}$ M1

Use of 'integration by parts' formula in the correct direction

$\text{I} = \frac{(x-1)^2}{2}\ln x - \int \frac{(x-1)^2}{2x} dx$ A1

Correct expression

$= \frac{(x-1)^2}{2}\ln x - \int \frac{x^2 - 2x + 1}{2x} dx$

$= \frac{(x-1)^2}{2}\ln x - \int \left(\frac{1}{2}x - 1 + \frac{1}{2x}\right) dx$

Candidate multiplies out numerator to obtain three terms...

... multiplies at least one term through by $\frac{1}{x}$ *and then attempts to ...*

... integrate the result; M1

<u>correct</u> integration A1

$= \frac{(x-1)^2}{2}\ln x - \left(\frac{x^2}{4} - x + \frac{1}{2}\ln x\right)(+c)$

$\therefore \text{I} = \left[\frac{(x-1)^2}{2}\ln x - \frac{x^2}{4} + x - \frac{1}{2}\ln x\right]_1^3$

$= (2\ln 3 - \frac{9}{4} + 3 - \frac{1}{2}\ln 3) - (0 - \frac{1}{4} + 1 - 0)$ ddM1

Substitutes limits of 3 and 1 and subtracts.

$= 2\ln 3 - \frac{1}{2}\ln 3 + \frac{3}{4} + \frac{1}{4} - 1 \ = \underline{\frac{3}{2}\ln 3}$ **AG** A1 cso 6

Aliter Way 4

(d) By substitution

$u = \ln x \qquad \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\mathrm{I} - \int (e^u - 1).ue^u \, du$

Correct expression

$= \int u\,(e^{2u} - e)\,du$ M1

Use of 'integration by parts' formula in the correct direction

$= u\left(\frac{1}{2}e^{2u} - e^u\right) - \int \underline{\left(\frac{1}{2}e^{2u} - e^u\right)} dx$ A1

Correct expression

$= u\left(\frac{1}{2}e^{2u} - e^u\right) - \underline{\left(\frac{1}{4}e^{2u} - e^u\right)}(+c)$

Attempt to <u>integrate</u>; M1;

<u>correct integration</u> A1

$\therefore l = \left[\frac{1}{2}ue^{2u} - ue^u - \frac{1}{4}e^{2u} + e^u\right]_{\ln 1}^{\ln 3}$

$= \left(\frac{9}{2}\ln 3 - 3\ln 3 - \frac{9}{4} + 3\right) - (0 - 0 - \frac{1}{4} + 1)$ ddM1

Substitutes limits of ln3 and ln1 and subtracts.

$= \frac{3}{2}\ln 3 + \frac{3}{4} + \frac{1}{4} - 1 \ = \underline{\frac{3}{2}\ln 3}$ **AG** A1 cso 6

$\frac{3}{2}\ln 3$

[13]

8. (a) $f'(x) = (x^2 + 1) \times \frac{1}{x} + \ln x \times 2x$ M1 A1

$f'(e) = (e + 1) \times \frac{1}{e} + 2e = 3e + \frac{1}{e}$ M1 A1 4

(b) $(\frac{x^3}{3} + x)\ln x - \int (\frac{x^3}{3} + x)\frac{1}{x}dx$ M1 A1

$= (\frac{x^3}{3} + x)\ln x - \int (\frac{x^3}{3} + 1)dx$

$= \left[(\frac{x^3}{3} + x)\ln x - (\frac{x^3}{9} + x)\right]_1^e$ A1

$= \frac{2}{9}e^3 + \frac{10}{9}$ M1 A1 5

[9]

9.	Attempts $V = \pi \int x^2 e^{2x} \mathrm{d}x$		M1	
	$= \pi \left[\frac{x^2 e^{2x}}{2} - \int x e^{2x} \mathrm{d}x\right]$	(M1 needs parts in the correct direction)	M1 A1	
	$= \pi \left[\frac{x^2 e^{2x}}{2} - \int x e^{2x} \mathrm{d}x\right]$	(M1 needs second application of parts)	M1 A1ft	
	M1A1ft refers to candidates $\int x e^{2x} \mathrm{d}x$, but dependent on prev. M1			
	$= \pi\left[\frac{x^2 e^{2x}}{2} - \left(\frac{x e^{2x}}{2} - \int \frac{e^{2x}}{4}\right)\right]$		A1 cao	
	Substitutes limits 3 and 1 and subtracts to give... [dep. on second and third Ms]		dM1	
	$= \pi \left[\frac{13}{4} e^6 - \frac{1}{4} e^2\right]$ or any correct exact equivalent.		A1	8
	[Omission of π loses first and last marks only]			

[8]

10. (a)	Attempt at integration by parts, i.e. $kx \sin 2x \pm \int k \sin 2x \mathrm{d}x$, with $k = 2$ or ½		M1	
	$= \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, \mathrm{d}x$		A1	
	Integrates sin 2x correctly, to obtain $\frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$		M1, A1	4
	(penalise lack of constant of integration first time only)			
(b)	Hence method: Uses $\cos 2x = 2\cos^2 x - 1$ to connect integrals Obtains		B1	
	$\int x \cos^2 x \mathrm{d}x = \frac{1}{2}\{\frac{x^2}{2} + \text{answer to part}(a)\} = \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + k$		M1A1	3
	Otherwise method			
	$\int x \cos^2 x \mathrm{d}x = x\left(\frac{1}{4}\sin 2x + \frac{x}{2}\right) - \int \frac{1}{4}\sin 2x + \frac{x}{2} \mathrm{d}x$		B1, M1	
	B1 for $(\frac{1}{4}\sin 2x + \frac{x}{2})$			
	$= \frac{x^2}{4} + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + k$		A1	3

[10]

11.	(a)	$\mathrm{I} = \int x \operatorname{cosec}^2(x + \frac{\pi}{6})\mathrm{d}x = \int x \mathrm{d}(-\cot(x + \frac{\pi}{6}))$	M1	
		$= -x\cot(x + \frac{\pi}{6}) + \int \cot(x + \frac{\pi}{6})\mathrm{d}x$	A1	
		$= \underline{-x\cot(x+\frac{\pi}{6}) + \ln(\sin(x+\frac{\pi}{6}) / c \ (*)}$	A1c.s.o.	3
	(b)	$\int \frac{1}{y(1+y)}\mathrm{d}x = \int 2x\operatorname{cosec}^2(x+\frac{\pi}{6})\mathrm{d}x$	M1	
		LHS = $\int\left(\frac{1}{y} - \frac{1}{1+y}\right)\mathrm{d}y$	M1 A1	
		$\therefore \ln y - \ln\lvert 1+y\rvert$ or $\ln\left\lvert\frac{y}{1+y}\right\rvert = 2(\mathrm{a})$	M1 M1	
		$\therefore \underline{\frac{1}{2}\ln\left\lvert\frac{y}{1+y}\right\rvert = -x\cot\left(x+\frac{\pi}{6}\right) + \ln\left\lvert\sin\left(x+\frac{\pi}{6}\right)\right\rvert + c\ (*)}$	A1c.s.o.	6
	(c)	$y = 1, x = 0 \Rightarrow \frac{1}{2}\ln\frac{1}{2} = \ln\left(\sin\frac{\pi}{6}\right) + c$	M1	
		$\therefore c = -\frac{1}{2}\ln\frac{1}{2}$	A1 M1	
		$x = \frac{\pi}{12} \Rightarrow \frac{1}{2}\ln\left\lvert\frac{y}{1+y}\right\rvert = -\frac{\pi}{12}.1 + \ln\frac{1}{\sqrt{2}} - \frac{1}{2}\ln\frac{1}{2}$	A1	
		$\sqrt{\text{"}c\text{"}}$		
		(i.e. $\ln\left\lvert\frac{y}{1+y}\right\rvert = -\frac{\pi}{6}$)		
		$\frac{y+1}{y} = e^{\frac{\pi}{6}}$	M1	
		(o.e.)		
		$1 = y(e^{\frac{\pi}{6}} - 1)$		
		$\therefore y = \underline{\frac{1}{e^{\frac{\pi}{6}} - 1}}$	A1	6
		(o.e.)		

[15]

12. Separating the variables

$\int \frac{dy}{y} = \int x^2 \cos x \, dx$ M1

LHS = ln y B1

RHS = $x^2 \sin x - \int 2x \sin x \, dx$ M1 A1 A1

$= x^2 \sin x - 2[x(-\cos x) - \int 1(-\cos x)\, dx\,]$ A1

$= x^2 \sin x + 2x \cos x - 2 \sin x + c$ A1

$y = 1$ at $x = \pi$ gives

$0 = 0 + 2\pi(-1) - 0 + c$

$\Rightarrow \quad c = 2\pi$

$\ln y = x^2 \sin x + 2x \cos x - 2 \sin x + 2\pi$ M1 A1

[9]

13. (a) $R = \int_{\pi}^{2\pi} x^2 \sin\left(\frac{1}{2}x\right) dx = -2x^2 \cos\left(\frac{1}{2}x\right) + \int 4x \cos\left(\frac{1}{2}x\right) dx$ M1 A1

$= -2x^2 \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) - \int 8\sin\left(\tfrac{1}{2}x\right)$ M1 A1

$= -2x^2 \cos\left(\frac{1}{2}x\right) + 8x \sin\left(\frac{1}{2}x\right) + 16 \cos\left(\tfrac{1}{2}x\right)$ A1

Use limits to obtain $[8\pi^2 - 16] - [8\pi]$ M1 A1 7

(b) Requires 11.567 B1 1

(c) (i) Area = $\frac{\pi}{4}$, [9.8696 + 0 + 2 × 15.702]

(B1 for $\frac{\pi}{4}$ *in (i) or* $\frac{\pi}{8}$ *in (ii))* B1, M1

= 32.42 A1

(ii) Area = $\frac{\pi}{8}$ [9.8696 + 0 + 2(14.247 + 15.702 + 11.567)] M1

= 36.48 A1 5

[13]

1. Candidates tended either to get part (a) fully correct or make no progress at all. Of those who were successful, most replaced the $\cos^2 \theta$ and $\sin^2 \theta$ directly with the appropriate double angle formula. However many good answers were seen which worked successfully via $7 \cos^2 \theta - 3$ or $4 - 7 \sin^2 \theta$.

Part (b) proved demanding and there were candidates who did not understand the notation $\theta\text{f}(\theta)$. Some just integrated $\text{f}(\theta)$ and others thought that $\theta\text{f}(\theta)$ meant that the argument 2θ in $\cos 2\theta$ should be replaced by θ and integrated $\frac{1}{2}\theta + \frac{7}{2}\cos\theta$. A few candidates started by writing $\int\theta\text{f}(\theta)\text{d}\theta = \theta\int\text{f}(\theta)\text{d}\theta$, treating θ as a constant. Another error seen several times was $\int \theta\,\text{f}(\theta)\text{d}\theta = \int\left(\frac{1}{2}\theta + \frac{7}{2}\cos 2\theta^2\right)\text{d}\theta$.

Many candidates correctly identified that integration by parts was necessary and most of these were able to demonstrate a complete method of solving the problem. However there were many errors of detail, the correct manipulation of the negative signs that occur in both integrating by parts and in integrating trigonometric functions proving particularly difficult. Only about 15% of candidates completed the question correctly.

2. Nearly all candidates gained both marks in part (a). As is usual, the main error seen in part (b) was finding the width of the trapezium incorrectly. There were fewer errors in bracketing than had been noted in some recent examinations and nearly all candidates gave the answer to the specified accuracy. The integration by parts in part (c) was well done and the majority of candidates had been well prepared for this topic.

Some failed to simplify $\int\frac{x^2}{2}\times\frac{1}{x}\text{d}x$ to $\int\frac{x}{2}\text{d}x$ and either gave up or produced $\frac{\frac{1}{3}x^3}{x^2}$.

In evaluating the definite integral some either overlooked the requirement to give the answer in the form $\frac{1}{4}(a1n\,2 + b)$ or were unable to use the appropriate rule of logarithms correctly.

3. Throughout this question sign errors were particularly common. In part (a), nearly all recognised that $(5-x)^{\frac{3}{2}}$ formed part of the answer, and this gained the method mark, but $\frac{3}{2}(5-x)^{\frac{3}{2}}, -\frac{3}{2}(5-x)^{\frac{3}{2}}$ and $\frac{2}{3}(5-x)^{\frac{3}{2}}$, instead of the correct $-\frac{2}{3}(5-x)^{\frac{3}{2}}$, were all frequently seen. Candidates who made these errors could still gain 3 out of the 4 marks in part (b)(i) if they proceeded correctly. Most candidates integrated by parts the “right way round” and were able to complete the question. Further sign errors were, however, common.

4. In part (a), a surprisingly large number of candidates did not know how to integrate $\tan^2 x$. Examiners were confronted with some strange attempts involving either double angle formulae or logarithmic answers such as $\ln(\sec^2 x)$ or $\ln(\sec^4 x)$. Those candidates who realised that the needed the identity $\sec^2 x = 1 + \tan^2 x$ sometimes wrote it down incorrectly.

Part (b) was probably the best attempted of the three parts in the question. This was a tricky integration by parts question owing to the term of $\frac{1}{x^3}$, meaning that candidates had to be especially careful when using negative powers. Many candidates applied the integration by parts formula correctly and then went on to integrate an expression of the form $\frac{k}{x^3}$ to gain 3 out of the 4 marks available. A significant number of candidates failed to gain the final accuracy mark owing to sign errors or errors with the constants α and β in $\frac{\alpha}{x^2}\ln x + \frac{\beta}{x^2} + c$. A minority of candidates applied the by parts formula in the 'wrong direction' and incorrectly stated that $\frac{dv}{dh} = \ln x$ implied $v = \frac{1}{x}$.

In part (c), most candidates correctly differentiated the substitution to gain the first mark. A significant proportion of candidates found the substitution to obtain an integral in terms of u more demanding. Some candidates did not realise that e^{2x} and e^{3x} are $(e^x)^2$ and $(e^x)^3$ respectively and hence $u^2 - 1$, rather than $(u - 1)^2$ was a frequently encountered error seen in the numerator of the substituted expression. Fewer than half of the candidates simplified their substituted expression to arrive at the correct result of $\int \frac{(u-1)^2}{u}\,du$. Some candidates could not proceed further at this point but the majority of the candidates who achieved this result were able to multiply out the numerator, divide by u, integrate and substitute back for u. At this point some candidates struggled to achieve the expression required. The most common misconception was that the constant of integration was a fixed constant to be determined, and so many candidates concluded that $k = -\frac{3}{2}$. Many candidates did not realise that $-\frac{3}{2}$ when added to c combined to make another arbitrary constant k.

5. In part (a), many candidates were able to use integration by parts in the right direction to produce a correct solution. Common errors included integrating e incorrectly to give $\ln x$ or applying the by parts formula in the wrong direction by assigning u as e^x to be differentiated and $\frac{dv}{dx}$ as x to be integrated.

Many candidates were able to make a good start to part (b), by assigning u as x^2 and $\frac{dv}{dx}$ and e^x again correctly applying the integration by parts formula. At this point, when faced with integrating $2xe^x$, some candidates did not make the connection with their answer to part (a) and made little progress, whilst others independently applied the by parts formula again. A significant proportion of candidates made a bracketing error and usually gave an incorrect answer of $e^x(x^2 - 2x - 2) + c$.

In part (b), a few candidates proceeded by assigning u as x and $\frac{dv}{dx}$ as xe^x and then used their answer to part (a) to obtain v. These candidates were usually produced a correct solution.

6. In part (a), many candidates recognised that the correct way to integrate $x \cos 2x$ was to use integration by parts, and many correct solutions were seen. Common errors included the incorrect integration of cos $2x$ and sin $2x$; the incorrect application of the 'by parts' formula even when the candidate had quoted the correct formula as part of their solution; and applying the by parts formula in the wrong direction by assigning $\frac{\mathrm{d}v}{\mathrm{d}x}$ as x to be integrated.

In part (b), fewer than half of the candidates deduced the connection with part (a) and proceeded by using "Way 1" as detailed in the mark scheme. A significant number of candidates used integration by parts on $\int x\left(\frac{\cos 2x+1}{2}\right)$ and proceeded by using "Way 2" as detailed in the mark scheme.

In part (b), the biggest source of error was in the rearranging and substituting of the identity into the given integral. Some candidates incorrectly rearranged the cos $2x$ identity to give $\cos^2 x = \frac{\cos 2x - 1}{2}$. Other candidates used brackets incorrectly and wrote $\int x\cos^2 x\mathrm{d}x$ as either $\int\left(\frac{x}{2}\cos 2x+1\right)\mathrm{d}x$ or $\int\left(\frac{x}{2}\cos 2x+\frac{1}{2}\right)\mathrm{d}x$.

A significant number of candidates omitted the constant of integration in their answers to parts (a) and (b). Such candidates were penalised once for this omission in part (a).

7. In part (a), the first mark of the question was usually gratefully received, although for x 1.5 it was not uncommon to see $\frac{1}{2}\ln(\frac{1}{2})$.

In part (b), it was not unusual to see completely correct solutions but common errors included candidates either stating the wrong width of the trapezia or candidates not stating their final answer correct to four significant figures.

Answers to part (c) were variable and often the mark in this part was not gained.

In part (d) all four most popular ways detailed in the mark scheme were seen. For weaker candidates this proved a testing part. For many candidates the method of integration by parts provided the way forward although some candidates applied this formula in the 'wrong direction' and incorrectly stated that $\frac{dv}{dx}$ =ln x implied v=^{1}. Sign errors were common in this part, eg: the incorrect statement of $\int\left(\frac{x}{2}-1\right)dx = -\frac{x^2}{4} - x$, and as usual, where final answers have to be derived, the last few steps of the solution were often not convincing.

In summary, this question proved to be a good source of marks for stronger candidates, with 12 or 13 marks quite common for such candidates; a loss of one mark was likely to have been in part (c).

8. The product rule was well understood and many candidates correctly differentiated f(x) in part (a). However, a significant number lost marks by failing to use ln e = 1 and fully simplify their answer.

Although candidates knew that integration by parts was required for part (b), the method was not well understood with common wrong answers involving candidates mistakenly suggesting that $\int \ln x\,\mathrm{d}x = \frac{1}{x}$ and attempting to use $u = x^2 + 1$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \ln x$ in the formula $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$.

Candidates who correctly gave the intermediate result $\left[\left(\frac{x^3}{3}+x\right)\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}}\left(\frac{x^3}{3}+x\right)\frac{1}{x}\,\mathrm{d}x$ often failed to use a bracket for the second part of the expression when they integrated and went on to make a sign error by giving $-\frac{x^3}{9}+x$ rather than $-\frac{x^3}{9}-x$.

9. There were many excellent solutions to this question but also too many who did not know the formula for finding the volume of the solid. Candidates who successfully evaluated $\int_1^3 x^2\mathrm{e}^{2x}\,dx$ were able to gain 6 of the 8 marks, even if the formula used was $k\int y^2\mathrm{d}x$ with $k \neq \pi$, but there were many candidates who made errors in the integration, ranging from the slips like sign errors and numerical errors to integrating by parts "in the wrong direction". An error with serious consequences for most who made it was to write $(x\, e^x)^2$ as $x^2\ e^{x^2}$; for some it was merely a notational problem and something could be salvaged but for most it presented a tricky problem !

10. (a) This was a straightforward integration by parts, which was recognised as such and done well in general. The most common error was the omission of the constant of integration, but some confused signs and others ignored the factors of two.

(b) This was done well by those students who recognised that $\cos^2 x = (1 + \cos 2x)/2$ but there was a surprisingly high proportion who were unable to begin this part. Lack of care with brackets often led to errors so full marks were rare. There was also a large proportion of candidates who preferred to do the integration by parts again rather than using their answer to (a).

11. Candidates found this question challenging; however those who read all the demands of the question carefully were able to score some marks, whilst quite an appreciable minority scored them all. In part (a), the crucial step involved keeping signs under control. Seeing

$$-x\cot\left(x+\frac{x}{6}\right) - \int -\cot\left(x+\frac{x}{6}\right)\mathrm{d}x$$

or a correct equivalent, demonstrated to examiners a clear method. Sign confusions sometimes led to a solution differing from the printed answer.

Part (b) was the main source for the loss of marks in this question. It was disappointing that so many candidates rushed through with barely more than three lines of working between separating the variables and quoting the printed answer, losing the opportunity to demonstrate their skills in methods of integration. The majority separated the variables correctly. Very few made any attempt to include the critical partial fractions step, merely stating

$$\frac{1}{2}\int\frac{1}{y(y+1)}\mathrm{d}y = \frac{1}{2}\ln\left(\frac{y}{1+y}\right)$$

as printed. Some did not recognise the right hand side of their integral related to part (a), producing copious amounts of working leading to nowhere.

In part (c), the working to evaluate the constant c was often untidy and careless. Those who persevered to a stage of the form $\ln P = Q + R$ generally were unable to move on to $P = \mathrm{e}^{Q+R}$ in a satisfactory manner, often writing $P = \mathrm{e}^Q + \mathrm{e}^R$.

12. No Report available for this question.

13. No Report available for this question.